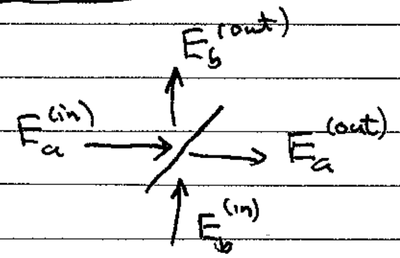
Physics 566

Problem Set #7: Solutions

The beam splitter and other linear transformations



$$\begin{bmatrix} E_a^{(out)} \\ E_b^{(out)} \end{bmatrix} = \underbrace{\begin{bmatrix} t & r \\ r & t \end{bmatrix}}_{\text{"S-matrix" } S} \begin{bmatrix} E_a^{(in)} \\ E_b^{(in)} \end{bmatrix}$$

(a) Unitarity of the S-matrix: $S^\dagger S = \mathbb{1}$

$$S S^\dagger = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} t^* & r^* \\ r^* & t^* \end{bmatrix} = \begin{bmatrix} |t|^2 + |r|^2 & t r^* + t^* r \\ t^* r + t r^* & |t|^2 + |r|^2 \end{bmatrix}$$

$\Rightarrow \boxed{|t|^2 + |r|^2 = 1}$ $\quad Re(t r^*) = 0$

$\Rightarrow \boxed{\text{Arg}(t) - \text{Arg}(r) = \pm \pi/2}$

Let $T = |t|^2$

$\Rightarrow t = \sqrt{T}\, e^{i\phi_t} \qquad r = i\sqrt{1-T}\, e^{i\phi_t}$

ϕ_t depends on details on beamsplitter

for $\phi_t = 0$

$$\boxed{\begin{aligned} E_a^{(out)} &= \sqrt{T}\, E_a^{(in)} + i\sqrt{1-T}\, E_b^{(in)} \\ E_b^{(out)} &= \sqrt{T}\, E_b^{(in)} + i\sqrt{1-T}\, E_a^{(in)} \end{aligned}}$$

(b) Quantized mode: $E_a \Rightarrow \hat{a} \qquad E_b \Rightarrow \hat{b}$

Suppose no field is injected into port "b"

Classically $\quad E_a^{(out)} = \sqrt{T}\, E_a^{(in)}$

Quantum analog $\quad \hat{a}^{(out)} = \sqrt{T}\, \hat{a}^{(in)}$?

__No__ $\quad [\hat{a}^{(out)}, \hat{a}^{(out)\dagger}] = T\,[\hat{a}^{(in)}, \hat{a}^{(in)\dagger}] = T \leq 1$

(c) ~~So~~ the uncertainty principle would be __violated__.

The problem is that we allowed __attenuation of vacuum fluctuations__. Formally, we violated unitarity in the transformation between input and output:

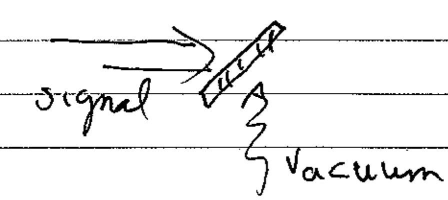
$$\hat{a}^{(out)} = \hat{S}^{\dagger} \hat{a}^{(in)} \hat{S} = \underbrace{\sqrt{T}}_{t}\, \hat{a}^{(in)} + \underbrace{i\sqrt{1-T}}_{r}\, \hat{b}^{(in)}$$

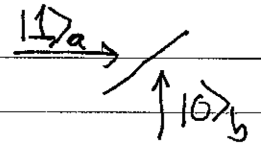
$$\Rightarrow [\hat{a}^{(out)}, \hat{a}^{(out)\dagger}] = |t|^2 [\hat{a}^{(in)}, \hat{a}^{(in)\dagger}] + |r|^2 [\hat{b}^{(in)}, \hat{b}^{(in)\dagger}]$$

$$+\, t r^* \underbrace{[\hat{a}^{(in)}, \hat{b}^{(in)\dagger}]}_{0} + t^* r \underbrace{[\hat{a}^{(in)\dagger}, \hat{b}^{(in)}]}_{0}$$

$$\Rightarrow \boxed{[\hat{a}^{(out)}, \hat{a}^{(out)\dagger}] = |t|^2 + |r|^2 = 1}$$

One way to interpret this is that although we do not input a signal into port-b, vacuum fluctuations __always__ enter that port



(d) 

Input single photon into mode-a and nothing in mode-b

$$\Rightarrow |\Psi_{in}\rangle = |1\rangle_a \otimes |0\rangle_b = \hat{a}^{(in)\dagger}|0,0\rangle \leftarrow \text{total vacuum}$$

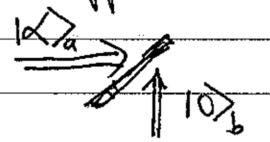
$$|\Psi_{out}\rangle = \hat{S}|\Psi_{in}\rangle = \hat{a}^{(out)\dagger}|0,0\rangle$$

$$= \left(t\,\hat{a}^{(in)\dagger} + r\,\hat{b}^{(in)\dagger}\right)|0,0\rangle$$

$$= t|1,0\rangle + r|0,1\rangle$$

$$\boxed{|\Psi_{out}\rangle = t|1\rangle_a \otimes |0\rangle_b + r\,|0\rangle_a \otimes |1\rangle_b}$$

(e) Now suppose we inject a coherent state



$$|\Psi_{in}\rangle = |\alpha\rangle_a \otimes |0\rangle_b$$

$$= \hat{D}_a^{(in)}(\alpha)\,|0,0\rangle$$

$\uparrow$ displacement operator)

$$\hat{D}_a^{(in)}(\alpha) \equiv \exp\{\alpha\,\hat{a}^{(in)} + \alpha^*\,\hat{a}^{(in)\dagger}\}$$

$$\Rightarrow |\Psi_{out}\rangle = \hat{S}|\Psi_{in}\rangle = \hat{S}\,\hat{D}_a^{(in)}\,\hat{S}^\dagger\,\underbrace{\hat{S}|0,0\rangle}_{=|0,0\rangle}$$

$$= \exp\{\alpha\,\hat{S}\hat{a}^{(in)}\hat{S}^\dagger + \alpha^*\,\hat{S}\hat{a}^{(in)\dagger}\hat{S}\}\;|0,0\rangle$$

$$= \exp\{\alpha\,\hat{a}^{(out)} + \alpha^*\,\hat{a}^{(out)\dagger}\}\,|0,0\rangle$$

$$= \exp\{\alpha(t\,\hat{a}^{(in)} + r\,\hat{b}^{(in)}) + \alpha^*(t^*\,\hat{a}^{(in)\dagger} + r^*\,\hat{b}^{(in)\dagger}\}$$

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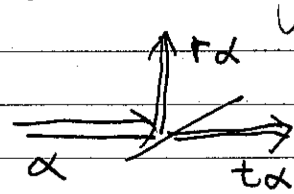
Since $\hat{a}^{(in)}$ and $\hat{b}^{(in)}$ modes commute

$$|\Psi_{out}\rangle = \hat{D}_a^{(in)}(t\alpha)\,\hat{D}_b^{(in)}(r\alpha)\,|0,0\rangle$$

$$\Rightarrow \boxed{|\Psi_{out}\rangle = |t\alpha\rangle_a \otimes |r\alpha\rangle_b}$$

This is the classically expected result

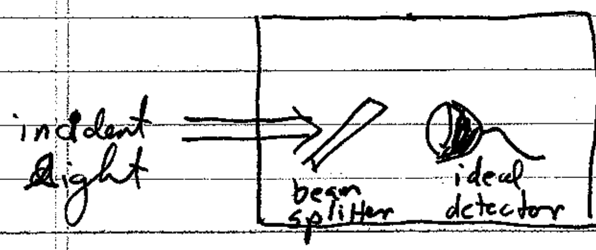


This is in contrast to the state

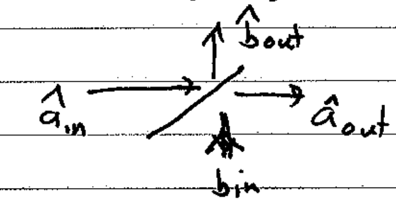
$$t|\alpha\rangle_a \otimes |0\rangle_b + r|0\rangle_a \otimes |\alpha\rangle_b$$

which is a "Schrödinger cat" state. This state describes a superposition of two macroscopic outcomes: the entire beam is transmitted (with probability $|t|^2$) or the entire beam is reflected (with probability $|r|^2$). This is a very nonclassical transformation, not accomplished by the linear beam splitter. The coherent state is basically a a many photon copy of the single photon state. Each photon acts independently and randomly takes the transmitted or reflected path. the Poisson Statistics are preserved.

(f) Model of an imperfect detector



The beam splitter acts to model the fact that a photon present in the field will only be detected with finite probability η.



$$\hat{a}_{out} = \sqrt{\eta}\,\hat{a}_{in} + i\sqrt{1-\eta}\,\hat{b}_{in}$$

The input field (consider pure state)

$$|\psi_{in}\rangle = \sum_n c_n |n_a\rangle \otimes |n_b\rangle = \sum_n c_n \frac{(\hat{a}_{in}^{\dagger})^n}{\sqrt{n!}} |0_a\rangle \otimes |0_b\rangle$$

The output field

$$|\psi_{out}\rangle = \sum_n c_n \frac{(\sqrt{\eta}\,\hat{a}^{\dagger} + i\sqrt{1-\eta}\,\hat{b}^{\dagger})^n}{\sqrt{n!}} |0_a\rangle \otimes |0_b\rangle$$

$$= \sum_{n,m} c_n \frac{(i)^{n-m}}{\sqrt{n!}} \binom{n}{m} (\eta)^{\frac{m}{2}} (1-\eta)^{n-m} (\hat{a}^{\dagger})^m (\hat{b}^{\dagger})^{n-m} |0\rangle$$

where $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$ is the binomial coeff

$$\Rightarrow |\psi_{out}\rangle = \sum_{n,m} c_n (i)^{n-m} \sqrt{\binom{n}{m} \eta^m (1-\eta)^{n-m}}\; |m\rangle_a \otimes |n-m\rangle_b$$

where I used $|m\rangle_a = \frac{(\hat{a}^{\dagger})^m}{\sqrt{m!}} |0\rangle_a$

$$|n-m\rangle_b = (\hat{b}^{\dagger})^{n-m} / \sqrt{(n-m)!}\; |0\rangle_b$$

We seek the probability of detecting m photons in the a-channel, irrespective of the number of photons in the b-channel.

$$\Rightarrow P_m = \sum_{n_b}' \left| \langle m_a, n_b | \psi_{out} \rangle \right|^2$$

$$= \sum_{n_a} |c_n|^2 \binom{n}{m} \eta^m (1-\eta)^{n-m}$$

$$\Rightarrow \boxed{P_m = \sum_n' P_n \underbrace{\binom{n}{m} \eta^m (1-\eta)^{n-m}}_{\text{Bernoulli distribution}}}$$

This expression has a simple interpretation. η is the probability of detecting one photon. Assuming statistically independent detections, given n photons the probability of detection m events is $\underbrace{(P(1 \text{ photon}))^m}_{\eta^m} \times \underbrace{\text{Probability of not detecting } n-m}_{(1-\eta)^{n-m}}$

There of $\binom{n}{m}$ different ways of choosing m out of n photons. The total probability given n photons $P(m|n) = \binom{n}{m} \eta^m (1-\eta)^{n-m}$

$$\Longrightarrow P_m = \sum_n P_n P(m|n) \quad \text{as above}$$

<u>Note</u>: Generally, the photo-electron statistics will not be a faithful reconstruction of the photon statistics. The exception is for classical light.

(g) Linear optics $\Rightarrow$ linear transformation of modes

$$E_k^{(out)} = \sum_{k'} \underbrace{\mathcal{U}_{kk'}}_{\text{unitary matrix}} E_{k'}^{(in)}$$

Quantum mechanically, $\hat{a}_k^{(out)} = \hat{S}\,\hat{a}_k^{(in)}\,\hat{S}^{\dagger} = \sum_{k'} \mathcal{U}_{kk'}\,\hat{a}_{k'}^{(in)}$

Suppose we start in an arbitrary multimode coherent state

$$|\psi_{in}\rangle = \hat{D}^{(in)}_{(\{\alpha_k\})}|0\rangle = \prod_k \hat{D}^{(in)}_{(\alpha_k)}|0\rangle$$

$$= \prod_k \exp\left(\alpha_k\,\hat{a}_k^{(in)\dagger} - \alpha_k^* \hat{a}_k^{(in)}\right) = \exp\left[\sum_k \left(\alpha_k \hat{a}_k^{(in)\dagger} - \alpha_k^* \hat{a}_k^{(in)}\right)\right]$$

$$|\psi_{out}\rangle = \hat{S}|\psi_{in}\rangle = \exp\left[\sum_k \left(\alpha_k \hat{a}_k^{(out)\dagger} - \alpha_k^* \hat{a}_k^{(out)}\right)\right]|0\rangle$$

$$= \exp\left[\sum_{k,k'}\left(\alpha_k \mathcal{U}^*_{kk'}\,\hat{a}_{k'}^{(in)\dagger} - \alpha_k^* \mathcal{U}_{kk'}\,\hat{a}_{k'}^{(in)}\right)\right]|0\rangle$$

$$= \exp\left[\sum_{k'}\left\{\left(\sum_k \mathcal{U}_{k'k}\,\alpha_k\right)\hat{a}_{k'}^{(in)\dagger} - \left(\sum_k \mathcal{U}_{k'k}\,\alpha_k\right)^* \hat{a}_{k'}^{(in)}\right\}\right]|0\rangle$$

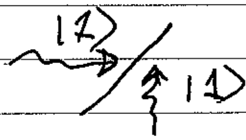
where I have used $\mathcal{U}^*_{kk'} = \mathcal{U}_{k'k}$ for unitary matrix

$$\Rightarrow \quad |\psi_{out}\rangle = \exp\left[\sum_{k'}\left(\tilde{\alpha}_{k'}^{(out)}\,\hat{a}_{k'}^{(in)\dagger} - \tilde{\alpha}_{k'}^{(out)}\,\hat{a}_{k'}^{in}\right)\right]|0\rangle$$

$$\boxed{\begin{gathered}\text{where } \tilde{\alpha}_{k'} = \sum_k \mathcal{U}_{k'k}\,\alpha_k \\ \Rightarrow \quad |\psi_{out}\rangle = \hat{D}(\{\tilde{\alpha}_k\})|0\rangle\end{gathered}}$$

(h) A non-classical input leads to nonclassical phenomena, even for linear transformations

Suppose $|\psi^{(in)}\rangle = |1\rangle_a \otimes |1\rangle_b$, two "mode matched" single photons incident simultaneously on a beam splitter:



$$\Rightarrow |\psi^{(in)}\rangle = \hat{a}^\dagger_{in}\hat{b}^\dagger_{in}|0\rangle$$

$$\Rightarrow |\psi^{(out)}\rangle = \hat{a}^\dagger_{out}\hat{b}^\dagger_{out}|0\rangle = \frac{1}{\sqrt{2}}(\hat{a}^\dagger - i\hat{b}^\dagger)(\hat{b}^\dagger - i\hat{a}^\dagger)|0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(a^\dagger b^\dagger - i a^{\dagger 2} - i b^{\dagger 2} + (-i)^2 b^\dagger a^\dagger\right)|0\rangle$$

overall phase

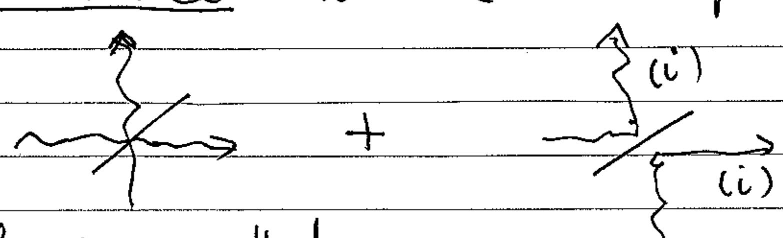
$$= \frac{-i}{\sqrt{2}}(\hat{a}^{\dagger 2} + \hat{b}^{\dagger 2})|0\rangle + \underbrace{\cancel{\frac{1}{\sqrt{2}}(a^\dagger b^\dagger - \hat{b}^\dagger \hat{a}^\dagger)|0\rangle}}_{\to 0}$$

since $[a^\dagger, b^\dagger] = 0$

$$\Rightarrow \boxed{|\psi^{(out)}\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle_a \otimes |0\rangle_b + |0\rangle_a \otimes |2\rangle_b\right)}$$

Thus both photons go off together.

We see that there is destructive interference for the two processes below

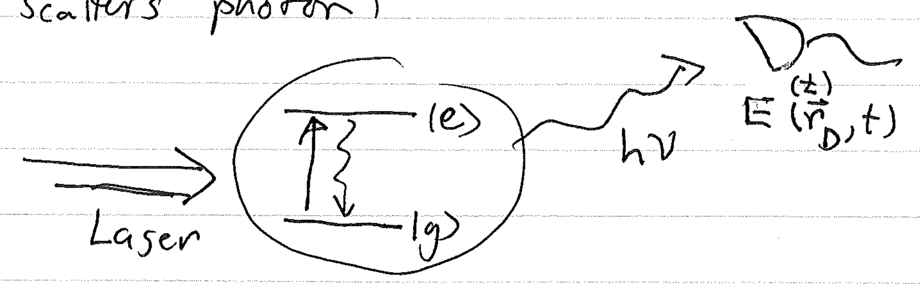


both transmitted

both reflected. Each picks up $\frac{\pi}{2}$ phase shift, causing destructive interference.

Problem 2: Resonance fluorescence

Two-level atom driven by classical field fluorescence (i.e. scatters photon)



$$E^{(\pm)}(\vec{r}_D, t) \propto \hat{\sigma}_{\mp}\left(t - \frac{r_D}{c}\right)$$

(a) Detected intensity

$$I(t) = \langle \hat{E}^{(-)}(\vec{r}_D, t)\, \hat{E}^{(+)}(\vec{r}_D, t)\rangle \propto \left\langle \hat{\sigma}_+\left(t - \frac{r_D}{2}\right) \hat{\sigma}_-\left(t - \frac{r_D}{2}\right)\right\rangle$$

Aside: $\hat{\sigma}_+(t) = (|e\rangle\langle g|)(t)$ $\quad \hat{\sigma}_-(t) = (|g\rangle\langle e|)(t)$

"Heisenberg Picture operator"

$$\Rightarrow \hat{\sigma}_+(t)\hat{\sigma}_-(t) = (|e\rangle\langle e|)(t)$$

$$\therefore I(t) \propto \mathrm{Tr}\left(\rho(0)\, (|e\rangle\langle e|)\left(t - \frac{r_D}{c}\right)\right) = \langle e|\rho|e\rangle\left(t - \frac{r_D}{c}\right)$$

$$= P_e\left(t - \frac{r_D}{c}\right)$$: Probability for the atom to be in the excited state at retarded time.

This makes sense physically. The rate at which the detector sees photons @ time t, depends of the probability that the atom spontaneously decay at the retarded time $t - \frac{r_D}{c}$. Moreover, the probability that the atom decayed at that time is the probability that the atom was in $|e\rangle$ at that time. Immediately after, it is in $|g\rangle$.

(b) We can always write ~~the~~ an operator as its mean value plus fluctuations about the mean relative to some state

$$\hat{\sigma}_{\pm}(t-\tfrac{r_D}{c}) = \langle\hat{\sigma}_{\pm}(t-\tfrac{r_D}{c})\rangle + \delta\hat{\sigma}_{\pm}(t-\tfrac{r_D}{c})$$

$$\Rightarrow I(t) \propto \langle\hat{\sigma}_{+}(t-\tfrac{r_D}{c})\,\hat{\sigma}_{-}(t-\tfrac{r_D}{c})\rangle$$

$$= \Big\langle\Big(\langle\hat{\sigma}_{+}(t-\tfrac{r_D}{c})\rangle + \delta\hat{\sigma}_{+}(t-\tfrac{r_D}{c})\Big)\Big(\langle\hat{\sigma}_{-}(t-\tfrac{r_D}{c})\rangle + \delta\hat{\sigma}_{-}(t-\tfrac{r_D}{c})\Big)\Big\rangle$$

$$= \langle\hat{\sigma}_{+}(t-\tfrac{r_D}{c})\rangle\langle\hat{\sigma}_{-}(t-\tfrac{r_D}{c})\rangle + \langle\delta\hat{\sigma}_{+}(t-\tfrac{r_D}{c})\,\delta\hat{\sigma}_{-}(t-\tfrac{r_D}{c})\rangle$$

(Having used $\langle\delta\hat{\sigma}_{\pm}\rangle = 0$)

$$\Rightarrow \boxed{I(t) \propto |\langle\hat{\sigma}_{+}(t-\tfrac{r_D}{c})\rangle|^2 + \langle\delta\hat{\sigma}_{+}(t-\tfrac{r_D}{c})\,\delta\hat{\sigma}_{-}(t-\tfrac{r_D}{c})\rangle}$$

Interpretation: In classical scattering as studied earlier in the context of the Lorentz oscillator, the radiated field is proportional to the induced dipole moment. The mean induced dipole moment

$$\langle\hat{d}^{(\pm)}(t)\rangle \propto \langle\hat{\sigma}_{\mp}(t)\rangle$$

In steady state ($t\to\infty$), this oscillates @ the frequency of the laser and represents the "coherent" part of the source field. The intensity that is proportional to $\langle\delta\hat{\sigma}_{+}\delta\hat{\sigma}_{-}\rangle$ represents the radiated field arising from quantum fluctuations.

Now $\langle \hat{\sigma}_+ \rangle = u + i v$ ← components of the Bloch vector

$$\Rightarrow |\langle \hat{\sigma}_+ \rangle|^2 = u^2 + v^2$$

In steady-state, we found in lecture 5b

$$u^{s.s.} = \frac{-2\Delta}{\Omega} \frac{s}{1+s} \qquad v^{s.s.} = \frac{-\Gamma}{\Omega} \frac{s}{1+s}$$

where $s = \dfrac{\Omega^2/2}{\Delta^2 + \frac{\Gamma^2}{4}}$ (saturation parameter)

$$\Rightarrow (u^2+v^2)^{s.s} = \left(\frac{4\Delta^2 + \Gamma^2}{\Omega^2}\right) \frac{s^2}{(1+s)^2} = \frac{2}{s} \left(\frac{s^2}{(1+s)^2}\right)$$

$$\Rightarrow \boxed{\langle I \rangle_{coherent} \propto \frac{1}{2} \frac{s}{(1+s)^2}}$$

(c) The incoherent part of intensity

$$\langle I \rangle_{incoherent} = \langle I \rangle_{total} - \langle I \rangle_{coherent}$$

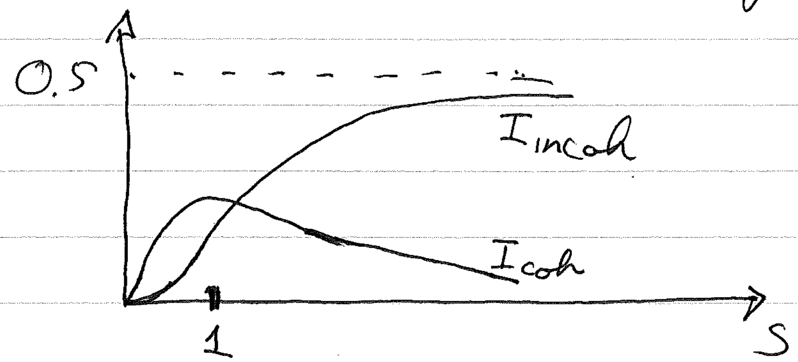
From part (a) $\langle I_{total} \rangle \propto P_e \left(t - \frac{r_0}{c}\right)$

In steady-state $P_e \rightarrow \dfrac{s}{2(1+s)}$ (Lecture #5b)

$$\therefore \boxed{\begin{aligned} \langle I \rangle_{incoherent} &\propto \frac{s}{2}\left[\frac{1}{1+s} - \frac{1}{(1+s)^2}\right] \\ &= \frac{1}{2} \frac{s^2}{(1+s^2)} \end{aligned}}$$

We see that $\dfrac{I_{incoherent}}{I_{coherent}} = S$

So for low saturation the scattering is predominantly coherent. We saw this in lecture #5b, where we showed that for $s \ll 1$ the response was like a linearly polarizable particle.



(Read about Mollow triplet!)

(d) We now consider the photon statistics associated with the scattered light. The two-photon correlation function

$$G^{(2)}(\tau) \propto \langle \hat{\sigma}_+(0)\, \hat{\sigma}_+(\tau)\, \hat{\sigma}_-(\tau)\, \hat{\sigma}_-(0) \rangle$$

$$= \mathrm{Tr}\left(\rho_0\, \hat{\sigma}_+(0)\, \hat{\sigma}_+(\tau)\, \hat{\sigma}_-(\tau)\, \hat{\sigma}_-(0) \right)$$

Here time $t=0$ is really long after transients have died off (i.e. steady state)

Again with $\hat{\sigma}_+(t) = (|e\rangle\langle g|)(t)$ $\hat{\sigma}_-(t) = (|g\rangle\langle e|)(t)$ and using cyclic property of trace:

$$\Rightarrow G^{(2)}(\tau) \propto \mathrm{Tr}\left((|e\rangle\langle e|)(\tau)\; |g\rangle\langle e|(0)\; \rho(0)\; |e\rangle\langle g|(0) \right)$$

Thus: $G^{(2)}(\tau) \propto \mathrm{Tr}\left((|e\rangle\langle g|)(\tau)\ |g\rangle\langle g|(0) \right) P_e(0)$

$$\boxed{G^{(2)}(\tau) \propto \mathrm{Tr}\left(\Pi_e(\tau)\, \Pi_g(0) \right) P_e(0)}$$

(e) Using the outer to inner product of trace

$$G^{(2)}(\tau) \propto |\langle e;\tau | g; 0\rangle|^2\ P_e(0)$$

$$\propto \underbrace{P(e;\tau | g;0)}_{\text{Conditional probability to be excited at } \tau \text{ given } g @ t=0}\ \underbrace{P_e(0)}_{\text{Prob to be in } e @ t=0}$$

We can understand the physical meaning of this result as follows. $G^{(2)}(\tau)$ measures the conditional probability of measuring a photon at time τ given a photon is detected at time $t=0$ (including retardation). This means that at $t=0$ (with retardation) the atom was in $|e\rangle$ and jumped to $|g\rangle$. Once in $|g\rangle$, the atom is excited again and if we see another photon @ τ the atom was in e @ τ (with retardation).

We can interpret this also as follows. After the first photon is detected, the atom is projected into $|g\rangle$. Then the atom is excited again.

(f) According to the optical Bloch equations, one can show that with the atom in $|g\rangle$ @ $t=0$

$$\langle\hat{\sigma}_z(t)\rangle = -1 + \frac{\Omega^2}{\Omega^2+\frac{\Gamma^2}{2}}\left(1 - e^{-\frac{3}{4}\Gamma t}\underbrace{\left(\cos\tilde{\Omega}t + \frac{3\Gamma}{4\tilde{\Omega}}\sin\tilde{\Omega}t\right)}\right)$$

where $\tilde{\Omega} \equiv \sqrt{\Omega^2 - \frac{\Gamma^2}{4}}$

$$\Rightarrow P(e;\tau|g,0) = \frac{1+\langle\hat{\sigma}_z(\tau)\rangle}{2} = \frac{\Omega^2/4}{\Omega^2+\frac{\Gamma^2}{2}}\left(\quad\downarrow\quad\right)$$

We get the normalized correlation function

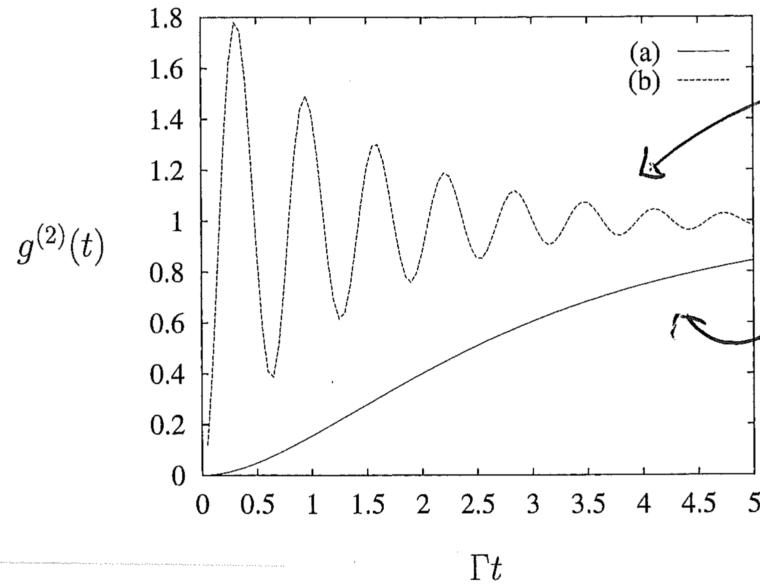
$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{I^2 \leftarrow \text{steady state intensity}} = \frac{G^{(2)}(\tau)}{(P_e(s.s.))^2}$$

On resonance $P_e(s.s.) = \dfrac{\Omega^2/4}{\Omega^2+\frac{\Gamma^2}{2}}$

$$\therefore \boxed{g^{(2)}(\tau) = 1 - e^{-\frac{3}{4}\Gamma\tau}\left(\cos(\tilde{\Omega}\tau) + \frac{3\Gamma}{2\tilde{\Omega}}\sin\tilde{\Omega}\tau\right)}$$

Note $g^{(2)}(0) = 0 \Rightarrow$ Photon antibunching for $\tau \ll 1$

(g)



high saturation (damped Rabi oscillations)

low saturation (overdamped) Rabi oscillations

We see here the photon antibunching. The time to detect a second photon, given one at $\tau = 0$ is zero. The time to see a second photon increases as the probability to excited the atom increases